[Based on a Symposium address by Dr Bill Pender]

The Excellence of the Present Calculus Courses

I want to talk first about the excellence of the present calculus courses. We mathematics teachers quietly think of them as the jewels in the crown of the NSW curriculum. Forty years of NSW students and teachers have been extremely fortunate to have them. My hope is that the Board leaves them alone, apart from some fine-tuning to clarify their structure. NSW should take notice of the serious loss of standards in some other States in recent years, and thank Professor Chong very much for his legacy, and Carslaw, Room, Brown, Cowling and many others.

The unity and coherence of these courses is extraordinary — everything links to everything else. The 2 Unit course is the basis of this unity. Beginning with sequences, it develops differentiation and integration, brings in $\sin x$ and $\cos x$, and e^x, gives some straightforward applications, and stops. That is the absolute kernel of calculus, a wonderfully tight structure, and 2 Unit achieves it.

I tell my classes that with $\sin x$ and e^x you can rule the world. Most of science, engineering, finance and economics is based on these two functions. This is because $\sin x$ describes fluctuating things, like seasons and vibrations, and e^x describes things that grow and decay exponentially, like inflation and radioactive decay.

There is a good mathematical reason behind this — you solve second-order differential equations with e^x and $\sin x$. When science throws up mathematics, it is mostly in the form of a differential equation, and so DEs fill the lecture halls of undergraduate science. DEs are rightly considered too tricky even for Extension 2, but DEs are just standard tools in most technical courses. Progression through many such courses is slowed down by the need to teach more mathematics.

Using the 2 Unit material in serious applications, however, requires a more sustained encounter with calculus, and with algebra. Extension 1 provides a firmer basis, and in doing so, achieves a new and more abstract unity. Many 2 Unit students later find they need more calculus, and re-enter university mathematics via a bridging course to Extension 1.

If you take any of the basic calculus kernel out of 2 Unit, such re-entry would be nearly impossible. Few General Mathematics students ever succeed in completing a degree in the physical sciences or engineering. Without the schoolroom's presentation of $\sin x$ and e^x, 2 Unit students would mostly be shut out of higher study in technical subjects. This important consideration should weight heavily on those deciding the future of the 2 Unit course.

Extension 2 completes the foundation study of calculus, and as we all know from recent examination papers, there is an amazing coherence in Extension 2. You can virtually pick any two fragments of the three calculus courses and relate them in some striking way.

The three calculus courses together are perfectly judged in terms of their standards. Anyone with the ability to learn calculus can make a decent fist of 2 Unit. Student headed for technical courses that will require university mathematics are well prepared by the Extension 1 course. And Extension 2, while attracting over 3000 candidates last year, still manages to challenge the very top students in each year's HSC to the limits of their abilities. Thus every student of calculus has a home in one of the three courses.

The applications of calculus are excellently chosen. Kinematics particularly is a wonderful application. Velocity makes the first derivative something you can see moving, acceleration makes the second derivative something you can feel as a car accelerates — it is not at all easy to find models for the second derivative suitable for school. Kinematics also provides a straightforward model for $\sin x$, while natural growth makes concrete the elusive abstraction of e^x, which 2 Unit students in particular find so difficult.

There is no need to seek out other applications in the calculus courses, particularly in the 2 Unit course. All our calculus students find applications of calculus very challenging because translating abstract variables into physical things is genuinely difficult. The present few, well-chosen, detailed and carefully-taught applications achieve far more than would a multiplicity of half-understood examples. In particular, further financial applications would not help — housing loans based on GPs have proven too challenging for 2 Unit, and anything further would probably involve approximating a GP by e^x, which would be even harder for 2 Unit.

But I return to my praise of our three calculus courses. These courses are imaginative and exciting for us to teach and for students to learn. The students at our school range from the top of Extension 2 to slightly below the bottom of 2 Unit, and in all three courses, they typically respond with a great sense of achievement as they work hard to master them. Extension 2 in particular generates amazing enthusiasm and is a wonderful foundation for university study at the highest level. We in mathematics are the envy of our colleagues, so many of whom have had to cope with unfortunate post-modernist trends that have undermined the language and structure of their disciplines — English and Physics are most often mentioned here. It would be sad to see calculus become merely a calculus appreciation course, as Physics now is said to be only a physics appreciation course.

University mathematics in NSW is in a good state at the moment, in contrast to many other States. The simple reason, as academics explain the situation to me, is the excellence of our HSC calculus courses. Given the competition from Southeast Asia, surely our task in schools is to maintain the clever country and develop it further.

Lastly, for an able student not continuing with mathematics, these are the right courses for an educated man or woman. They train students to think, and by teaching calculus, they provide a window into the nature of mathematical and scientific thought.

Problems in the 2 Unit Course

Recent problems in 2 Unit principally arise from actions by the universities. First, they dropped prerequisites, so increasing their numbers with students unlikely to achieve success. Secondly, they have allowed their UAI scaling to overvalue General Mathematics in comparison with 2 Unit, and there now seems to be a feedback loop, with many calculus-capable students opting for General, thus forcing up even further the university-scaling mean of General.

The effect has been to encourage able students away from proper mathematics to an HSC course that hardly extends their mathematics beyond Year 10. This situation needs urgent remedy by the restoration of honest prerequisites and the intervention of the scaling committee, otherwise 2 Unit could be stone dead well before the Board's present review is complete.

But the Board could help too. Its 2 Unit calculus examinations have become too hard for the candidature. Separating the top should not overwhelm other considerations. Here is a basic principle of all examining — diligent students of adequate ability should come out of the examination room feeling that they have given a good account of themselves.

The 2 Unit syllabus could be cleaner so that the basic structure of calculus, with $\sin x$ and e^x, stands out better. Some non-essential tricky details obscure this structure. Here is a short list of some obvious targets for removal:

- difference of cubes
- line through the intersection of two given lines (from Extension 1 as well)
- perpendicular distance formula
- housing loans
- differentiate $x \sin x$ and hence integrate $x \cos x$
- Simpson's rule (from Extensions 1 & 2 as well; the trapezoidal rule is sufficient)
- the words 'definite quadratic' and 'indefinite quadratic' (from Extensions 1 & 2 as well)

Locus and the parabola should go from 2 Unit (but not from Extension 1, where it provides a wonderful unity of algebra and geometry). Then the weak 2 Unit probability could be strengthened with a basic no-frills treatment of counting, including permutations and combinations. This would give 2 Unit students at least a toehold on the binomial distribution. I'm uneasy about 2 Unit graduates not being able to count the number of possible car number plates or the possible orderings of a queue.

Pathways restrictions should go. Recent HSC papers have had to concentrate on the much harder Year 12 topics, which is one reason why the examination papers have become too hard for our 2 Unit students. Pathways also causes endless programming headaches in schools.

It may be possible to decouple the Extension 1 students from the 2 Unit examinations. There would be many benefits from scaling the pure 2 Unit students as a candidature on their own.

Some Recommended Content Changes to the Calculus Courses

Here is a short list of content details that need attention.

- Transformations of functions have been gradually and successfully introduced by teachers over the last few years, but only the Extension 2 syllabus reflects this evolution. Translations, reflections in the axes, and rotation of 180° belong in the 2 Unit course. Stretching, and addition, subtraction and reciprocals of functions belong in Extension 1. General composition of functions is already in Extension 2, but reflections in other vertical and horizontal lines should be added.
- The function ${}^n\mathrm{C}_r$ in Extension 1 should be based on counting. The present definition of ${}^n\mathrm{C}_r$ as a coefficient in the expansion of $(x+y)^n$ has proven too abstract for school students.
- Simple harmonic motion in Extension 1 should be defined by the time function $a\sin(nt+\varepsilon)$ or $a\cos(nt+\varepsilon)$. Defining SHM as the solution of the DE $\ddot{x} = -n^2x$ is rather sophisticated for school, and the present situation of having two alternative definitions even confused the HSC examiners recently.
- The section in Extension 2 entitled 'Harder Extension 1 questions' should be written out in far more detail so that teachers and students know what is to be covered. The present situation gives a quite unfair advantage to schools with large numbers of candidates and highly qualified mathematics teachers.
- The expected value of a binomial variable should be restored to Extension 1. Expecting 10 sixes in 60 throws of a die is a basic part of binomial probability, and is not hard.

The Text of the Syllabus

The 2 Unit and Extension 1 syllabus documents themselves are poorly written, in four aspects:

- The mathematics is often unclear.
- The notation is often clumsy.
- It is often unclear what students should learn, or what level of treatment is required.
- There are often too many words, as in the discussions of continuity and of the fundamental theorem. But there are often too few words, as in the calculus of e^x.

Indeed the presentation of e^x in the calculus courses needs considerable attention.

The Board has made some rather artificial constraints on the structure of its syllabus documents. I would like to see the Board vary its constraints so as to allow the calculus syllabuses to be written as coherent pieces of mathematics. In effect, I am suggesting the syllabus look more like the theory from a standard course textbook.

I constantly meet fine teachers who say that they lack resources from which to learn what it is that the Board wants them to teach. Unfortunately, textbooks that fall into the subtle and unsubtle traps and errors of elementary calculus don't help either. A mathematics degree is excellent preparation, but it does not teach you directly how to teach elementary calculus so that it is elegant, imaginative and appropriately rigorous. This is the task that I believe the Board's Syllabus text should address.

That syllabus writing task will need to be led by academics who are real experts in analysis, working with experienced school teachers. There is no room for poor drafting in calculus.

What should not change — Euclidean geometry

In case it is again suggested that Euclidean geometry be cut from our calculus courses, I reiterate that mathematics rests on the two pillars of arithmetic and geometry. Arithmetic intuition rests on rhythms and counting. Geometric intuition rests on our visual field and tactile sense. All mathematics, from Greek times, seems to have thrived on the constant contrasts between these two sorts of intuition. In fact, it was only in the late 19th century that real numbers and the Euclidean plane were brought into logical harmony.

One of the great strengths of our calculus courses is the foregrounding of this interaction between geometry and arithmetic in the graphing of functions. The table of values is arithmetic — the shape of the curve is geometry. Calculus constantly uses algebra to solve geometric problems, and geometry to solve algebraic problems, just as in wider mathematics.

It is important that both geometry and arithmetic/algebra are each developed rigorously at school, sometimes independently, and sometimes combined. One without the other is not mathematics as it has ever been understood.

Mathematical reasoning is far easier to teach in geometry than in algebra, because of the pictures. Hence the traditional use of Euclidean geometry to teach mathematical reasoning in school. Although the 2 Unit course only extends geometry a little beyond Years 9–10, most students are not mature enough then to appreciate the proofs, and as is usual in teaching, they only really assimilate the material when it is revisited in Years 11–12.

What should not change — Technology

Introducing graphics calculators into the calculus examinations would wreck the calculus courses. I have no problems with technology — we all use computers every day. I tell my classes that to get on in life, they should be able to drive a car, programme a spreadsheet and play the piano. But graphics calculators are already years out of date. They were only ever intended for schools, by calculator companies specifically seeking a market. Universities and industry never took them up — they are a dead end. Why should the Board mandate such a clunky technology, when so many schools now have computer classrooms that leave graphics calculators for dead?

The Board should not mandate technology at all. Some teachers use it most effectively — good on them, and give them free rein! Other schools, however, like ours, are able to produce wonderful motivation and understanding better without technology than with. Let people teach as suits their students' situations and their own enthusiasms.

Graphics calculators in examinations would work against some of the key skills that we are teaching in calculus, that is, the recognition and graphing of curves. The courses unify much of their content around curve-sketching — if you start by getting the curve on the screen, you will kill this skill, and end up with dumbed-down calculus that is the bane of academics trying to teach calculus properly at tertiary level. NSW academics tell many stories of students from other States who were trained on graphics calculators and get into serious difficulties in university courses because they can't sketch simple curves or interpret them.

It's hard to take seriously the claim that graphics calculators develop higher-order skills without the burden of learning supposedly unimportant lower-order skills. How would 2 Unit students develop higher-order skills when they would not even be able to sketch $y = \dfrac{6}{x}$ or $y = e^{-x}$ or $y = \sin 3x$ without the calculator? There's an unreality about classroom teaching here.

Now there is talk of a graphics calculator that does algebra for you. From my point of view as a classroom teacher, that would be a complete disaster, with students not even learning properly how to expand and factor. There is no way that students can do or understand the heavy mathematics in technical tertiary courses without algebraic manipulation. 2 Unit students particularly need constant drill in straightforward algebra — slips like $(x+y)^2 = x^2+y^2$ are still common in the HSC.

We should learn from the mistakes made when scientific calculators were introduced. We know now that the place for scientific calculators is in Years 9–12, where they have greatly assisted our teaching of trigonometric and exponential functions. Misplaced enthusiasm, however, introduced them in Years K–8, when children were still busily learning arithmetic, and classroom teaching of arithmetic suffered badly as a result. Some educational theorists even compounded the damage by claiming that arithmetic was no longer necessary, and that children should no longer learn their tables. We should be similarly cautious of graphics calculators in Years 9–12, when children are first learning to graph functions, and where the table of values is constantly needed as a basis for their understanding. Graphics calculators should not be used routinely in class at this stage, and they certainly have no place in public examinations of these students' knowledge.

Technology is sometimes claimed to be desirable because it allows students to solve real-life problems instead of artificially simplified problems. Such arguments misunderstand how physics and mathematics have achieved such outstanding success with real-life problems over the last four centuries. They characteristically deal first with first-order effects, and analyse them completely, before considering second- and third-order effects. For example, when a compact object is thrown, the first-order effect is parabolic motion — that's in Extension 1. Air resistance may be a second-order effect, which with certain assumptions can still be solved analytically — that's in Extension 2. Then it may be possible to deal with tumbling and aerodynamic lift and so forth as perturbations on that motion. Without such a hierarchy of effects, you would not have Newton's laws of motion or gravity in physics, nor their relationship with theorems on conics in mathematics, you would just have streams of numbers. The purpose of school mathematics is to present the huge drama of simple and elegant structures in physics and mathematics — complexity doesn't teach beginners much at all, it only confuses the structures.

In summary, graphics calculators in examinations would move our calculus courses from their present role of providing solid training and foundation for academic work towards calculus appreciation courses, and hold up university teaching, just as the HSC Physics course has done. Recent comparisons of Year 12 mathematics courses across States have confirmed this view.

What should not change — Statistics

After Year 10, there is, unfortunately, no real way forward in mathematics itself for those who cannot cope with calculus. Unlike the calculus courses, however, General Mathematics is not working, because it is trying to cater for too many different abilities. Two courses are urgently needed, with as many chances of re-entry to calculus as is reasonably possible.

Applications should provide the structure of these courses. That means statistics, finance, spreadsheets and the like, on proper computers, not graphics calculators. I will leave these discussions to other, because I have no direct experience of teaching General.

But for calculus students, the overwhelming priority is to learn enough calculus so that a university can teach them how mathematics, including statistics, really works. Proper statistics is genuinely hard. It begins with the normal distribution, because of the central limit theorem, and the normal requires sound knowledge of both integration and e^{-x}, which comes only at the end of 2 Unit.

It would be absurd to put statistics into 2 Unit and throw out e^x — the normal distribution function has $e^{-\frac{1}{2}x^2}$ as its basic element! Bridging courses would become vastly more difficult, guaranteeing that the large numbers of 2 Unit students are cut off from tertiary courses requiring serious mathematics. Is statistics worth such a black hole in the 2 Unit syllabus?

Extensions 1 & 2 students do not need statistics at school — they need strong calculus as preparation to understand the statistics they will meet. We could, only just, introduce the normal to Extension 1, and the Poisson to Extension 2, but they are really hard, you can't do much with them at school, and there are better topics to train school students in mathematics.

Statistical tests do not fit into school calculus courses, because students can't possibly understand why they work — black-box methodology is contrary to the nature of mathematics. Again, such things are more in the nature of statistics appreciation, because they teach methods, but without the understanding needed in a course providing foundation for academic work.

A 1 Unit Year 12 Statistics course for 2 Unit students?

Black-box statistics is not mathematics — both mathematicians and statisticians seem to agree on this — they need separate courses. One solution would be a 1 Unit Year 12 statistics course for 2 Unit students, to satisfy those arguing that there is a gap. Such a course could be taught with totally different methodology — black-box statistical tests, heavy use of computers, whatever statisticians want. (By the way, they mostly want more calculus!)

What should not change — Linear algebra and matrices

This is my subject, but I would not teach it at school. All you can do is set up a piece of blank paper in n dimensions — doing anything with it is far too advanced for school. Freddy Chong was right — calculus is uniquely suited to school. It is more generally useful than other branches, and it provides a steady ascent into mathematics, not the sudden leaps of difficulty of discrete mathematics, linear algebra, statistics and abstract algebra.

UNSW is about to publish a study of the opinions of mathematics academics across all of the NSW and ACT Mathematics departments in relation to the current HSC syllabuses. Majority opinion strongly favours the present content, with little support for matrices or statistics, and even less for examination use of technology.

This is the place to mention APs and GPs. They are the one part of discrete mathematics that can be taught easily at school, and they allow many of the 'patterns' of K–10 mathematics to be placed in a more systematic context. Of course they are also an essential part of calculus — it would make little sense to teach e^x without having first taught 2^n, or to teach the integral as being intuitively 'an infinite sum of infinitesimals' without having taught the sum of a simple series.

What should not change — Assessment

Alternative assessment methods don't belong in calculus examinations. The courses must not move towards sociology, with projects, talks, posters, and essays on 'the effect of $\sin x$ on society and the environment'. The purpose of mathematics courses is to teach mathematics, because you need mathematics to understand the natural and human world around you, and because it is urgently needed at university.

The impressive thing about our HSC calculus examinations is that teaching students mathematics, and preparing them for the HSC examinations, are exactly the same task — a very different experience from our colleagues in other subjects. For many years now, the calculus examinations have been wonderful tests of problem solving and of understanding (some might say a few too many problems and a little too much understanding, particularly for 2 Unit students).

In particular, multiple choice is inappropriate in calculus examinations. It is a poor discriminator at this level and wastes examination time. The present structure of equal-value questions is excellent.

Professional Development

Professional development is badly needed. How many calculus teachers understand the importance of $\sin x$ and e^x in science and finance, and their role in DEs? How many of them understand the extraordinary unity of the courses they are teaching? Teachers are impressively hard-working people, but mathematics is difficult and they need time out of the classroom to reflect on their discipline, and financial recognition for having done so.

An unfortunate aspect of school teaching is that after teachers get their degree, they mostly spend the next 35 years or so teaching mathematics with no provision made whatsoever for any further serious study of mathematics. We would never tolerate such treatment of our doctors or engineers. Teachers need regular periods of time away from the classroom to take further mathematics courses and to deepen their understanding of the mathematics that they teach.

The progression to teaching Extension 2, however, has been an excellent spur for teachers to learn more mathematics. NSW should be proud of its many excellent mathematics teachers.

Summary of Recommendations

1. Create two non-calculus courses.
   Possibly create a 1 Unit Year 12 statistics course for 2 Unit students.

2. Maintain three levels of calculus courses.
   Maintain their content structure.
   Drop the remaining *Pathways* restrictions.

3. Rewrite the syllabus texts as coherent mathematics.
   Remove some non-essential details from 2 Unit, to make the basic structure clearer.
   Replace 2 Unit locus by simple, no-frills counting.
   Restructure some other topics in the three courses, as listed above.
   Maintain kinematics, sequences and Euclidean geometry.
   Maintain conics in Extensions 1 & 2.

4. Encourage appropriate modern technology in class.
   Don't introduce graphics calculators into examinations.
   Leave descriptive and black box statistics out of calculus.
   Be cautious about introducing the normal or Poisson.

5. Respect what our calculus courses have achieved.
   Value the knowledge and morale of our mathematics teachers.
   Be serious about developing the mathematical knowledge of teachers.

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